Optimization of Interior Point Cost Functionals Using Indirect Methods

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Conventional trajectory optimization methods focus on optimizing objectives defined over the whole trajectory or at the terminal point of the trajectory. Methods for solving cost functionals defined at an interior point have been described by Vinh and Powers in prior work. Such problems, also called minimax or maximin problems, involve the optimization of a cost functional that peaks at some arbitrary point along the trajectory. The method used to solve these problems depends on whether the cost functional forms an isolated peak or a flat peak. A numerical algorithm is developed that identifies the type of peak formed by a cost functional and automatically selects the right method to solve it. The algorithm uses behavior of the time-derivative of the cost functional to assess if it is better modeled as a flat peak or an isolated peak. The algorithm and reasoning that led to its design are demonstrated using two representative optimal control problems.

Nomenclature

$\boldsymbol{x}$	State vector	J^*	Augmented cost function
$\dot{q}$	Heating rate, rad	k	Sutton-Graves heating constant, $\sqrt{\text{kg}}/\text{m}$
D	Drag force, N	L	Lift force, N
H	Hamiltonian	m	Glider mass, kg
h	Altitude, m	R_E	Radius of the Earth, m
I	Interior point cost function	r_n	Nose radius, m
J	Cost function	v	Velocity, m/s
$\boldsymbol{\lambda}$	Costate vector	γ	Flight-path angle, rad
$\boldsymbol{\Phi}$	Terminal point constraints	ρ_0	Atmospheric density at surface, kg/m^3
$\boldsymbol{\Psi}$	Initial point constraints	θ	Downrange, rad

I. Introduction

Traditional trajectory optimization methods focus on the optimization of cost functionals evaluated at the terminal point, or a path cost evaluated over the entire trajectory. The optimization of cost

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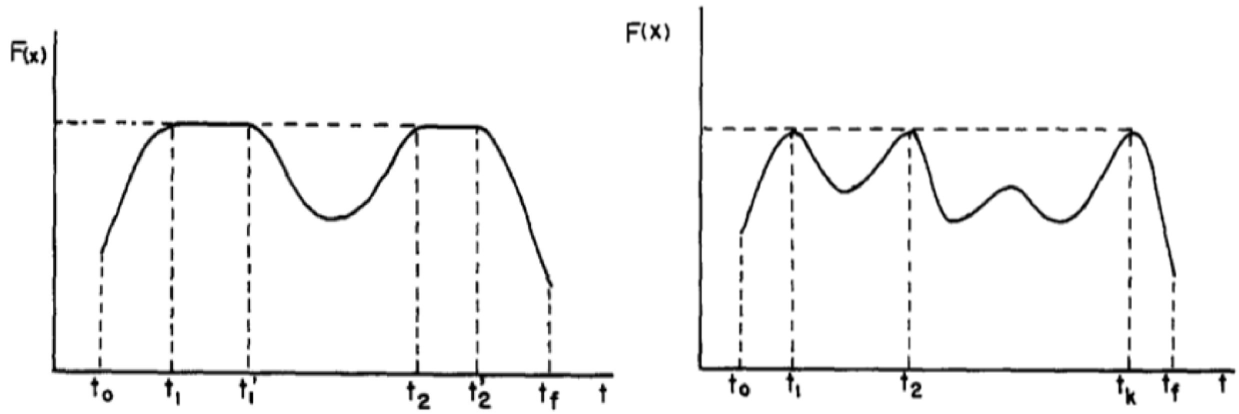
functionals defined at arbitrary intermediate points, also called minimax or maximin problems, have largely been ignored in literature in the last two decades. Vinh[1] and Powers[2] have derived the necessary conditions of optimality for unique, isolated peaks as well as flat maxima/minima (Fig. 1). Vinh and Lu developed semi-analytic solutions for a few aerospace problems using this method.[1,3]

However, their method requires some a-priori knowledge of the nature of the peak being optimized. We intend to automate the process of identifying the type of peak during the solution process and apply the appropriate conditions to optimize it. An example where this can be applied is for optimizing the peak heat rate or deceleration of a hypersonic vehicle on an entry trajectory. This peak can occur anywhere in the trajectory and the objective will be to minimize this peak value.

Modern direct trajectory optimization solvers such as GPOPS[4] focus entirely on solving terminal point cost functionals and path costs have to be reformulated in order to optimize them. Previous work[5] has shown that indirect optimization methods using calculus of variations, combined with continuation, allows fast computation of high quality solutions in an automated fashion. This allows the application of these methods to solve complex hypersonic trajectory problems, which was previously considered impractical.



(a) Multiple flat peaks ($q = 1$)

(b) Multiple isolated peaks ($q \geq 1$)

Figure 1. Multiple flat maxima or isolated maxima[1]

Vinh[1] identifies two different kinds of peaks in a trajectory that can be optimized (Fig. 1). There can be flat peaks in which the cost functional reaches a maximum and stays at that value for a finite interval of time as well as isolated peaks which end in a single point. The method to be used to solve the problem depends on the type of peak encountered in the trajectory.

Let the cost functional be of the form

$$\text{Min Max(or Min) } F(\boldsymbol{x}) \tag{1}$$

The type of peak encountered depends on the number of time-derivatives of $F(\boldsymbol{x})$, denoted by q, required before the control variable appears. If $q > 1$, the functional will form one or more isolated peaks as in Fig. 1b. Vinh's original work examines multiple cases where $q > 1$, but does not go into much detail about $q = 1$ cost functionals other than saying that they can have either flat or isolated peaks. This paper examines a method that can be used to identify the type of peak in a cost functional with $q = 1$ in order to solve the optimal control problem using indirect methods.

II. Necessary Conditions of Optimality

A. Isolated Peak

The necessary conditions of optimality assuming the minimization of a single, isolated peak is derived by applying optimal control theory.[6] The cost functional is applied at an interior point where the time-derivative of the cost functional is equal to zero. This is achieved by adding an interior point constraint to the problem.

The problem is stated as follows :

$$\begin{aligned}
&\text{Min } J = I(\boldsymbol{x}(t_1), t_1) \\
&\text{Subject to :} \\
&\dot{\boldsymbol{x}} = \boldsymbol{f}(x, u, t) && \text{(2a)} \\
&\boldsymbol{\Psi}(\boldsymbol{x}(t_0), t_0) = 0 && \text{(2b)} \\
&\boldsymbol{\Phi}(\boldsymbol{x}(t_f), t_f) = 0 && \text{(2c)} \\
&N(\boldsymbol{x}(t_1), t_1) = 0 && \text{(2d)} \\
&\text{where } N = \frac{dI}{dt} = \frac{\partial I}{\partial \boldsymbol{x}}^T \cdot \boldsymbol{f}
\end{aligned}$$

The augmented cost functional and hamiltonian are defined in Eqs.(3) and (4) respectively.

$$J^* = I(\boldsymbol{x}(t_1), t_1) + \boldsymbol{\nu_0}^T \boldsymbol{\Psi} + \boldsymbol{\nu_f}^T \boldsymbol{\Phi} + \pi N \tag{3}$$

$$H = \boldsymbol{\lambda}^T \boldsymbol{f} \tag{4}$$

where $\boldsymbol{\lambda}$ is the vector of costates,[6] $\boldsymbol{\nu_0}$, $\boldsymbol{\nu_f}$ and π are Lagrange multipliers used to adjoin the initial, terminal and interior point constraints. The necessary conditions of optimality are :

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}} \tag{5a}$$

$$\boldsymbol{\Psi}(\boldsymbol{x}(t_0), t_0) = 0 \tag{5b}$$

$$\boldsymbol{\Phi}(\boldsymbol{x}(t_f), t_f) = 0 \tag{5c}$$

$$\boldsymbol{\lambda(t_0)} = -\boldsymbol{\nu_0}^T \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{x}(t_0)} \tag{5d}$$

$$\boldsymbol{\lambda(t_f)} = \boldsymbol{\nu_f}^T \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}(t_f)} \tag{5e}$$

$$\boldsymbol{x}({t_1}^-) = \boldsymbol{x}({t_1}^+) \tag{5f}$$

$$\boldsymbol{\lambda({t_1}^+)} - \boldsymbol{\lambda({t_1}^-)} + \frac{\partial I}{\partial \boldsymbol{x}(t_1)} + \pi \frac{\partial N}{\partial \boldsymbol{x}(t_1)} = 0 \tag{5g}$$

$$H(t_f) = 0 \tag{5h}$$

$$H({t_1}^-) - H({t_1}^+) + \frac{\partial I}{\partial t_1} + \pi \frac{\partial N}{\partial t_1} = 0 \tag{5i}$$

When solving the problem using indirect methods, it can be difficult to provide an initial guess for the corner conditions. In order to avoid this, a continuation strategy is used where an unconstrained problem with a different objective functional is first solved. The interior point constraint described in Eq. (2d) is then added to this unconstrained trajectory and a weighted objectives method is used to change the problem from the original objective to the interior point cost functional of interest.

B. Flat Peak

Optimizing a flat peak is very similar to enforcing a path constraint on the trajectory. Vinh[1] describes a method which uses a dummy variable in order to directly find the minimum or maximum possible constraint limit. A cost functional of the form in Eq. (1) is used to define a new optimal control problem as follows:

$$\begin{aligned}
&\text{Min } J = x_{N+1}(t_f) \\
&\text{Subject to :} \\
&\dot{\boldsymbol{x}} = \boldsymbol{f}(x,u,t) && \text{(6a)}\\
&\frac{\boldsymbol{dx_{N+1}}}{\boldsymbol{dt}} = 0 && \text{(6b)}\\
&\boldsymbol{\Psi}(\boldsymbol{x}(t_0),t_0) = 0 && \text{(6c)}\\
&\boldsymbol{\Phi}(\boldsymbol{x}(t_f),t_f) = 0 && \text{(6d)}\\
&g(x) = (F(\boldsymbol{x}) - x_{N+1}) < (\text{or } >)\ 0 && \text{(6e)}
\end{aligned}$$

The necessary conditions of optimality are the same as those used to enforce a path constraint on the trajectory.[6] However, it is difficult to obtain a viable initial guess to solve the boundary value problem resulting from Vinh's approach. This problem can by overcome by using a continuation approach as in Section II.A where a different objective functional is optimized before the minimax problem is introduced. $F(x)$ is added as a simple path constraint to the problem and the minimum (or maximum) value of this constraint that can be enforced while still satisfying the other boundary conditions in the problem, is found. This is done by changing the constraint limits in a series of steps until the problem becomes infeasible.

The caveat in both of these continuation approaches is that they require a separate objective functional to be optimized first before the interior point cost functional can be introduced. This requirement is not a significant drawback since it is relatively uncommon to solve problems that contain only minimax cost functionals. Generally there are other objectives of interest and the minimax problem is used to determine the limits of potential constraints in the problem.

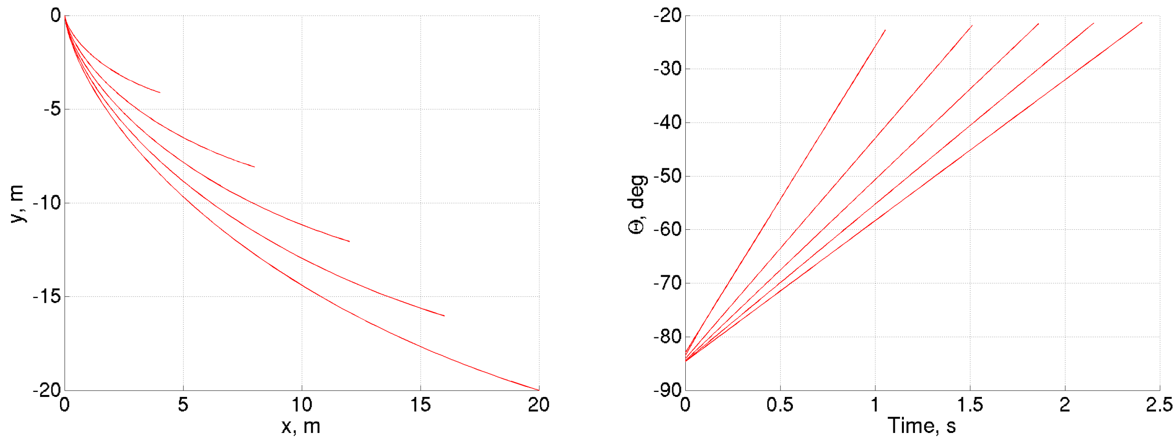
III. Test Scenarios for Analysis

A. Modified Brachistochrone Problem

The method is applied to solve a modified version of the brachistochrone problem.[6] This is an example of a cost functional with $q = 1$ that forms an isolated peak. The objective of the brachistochrone problem is to find the minimum-time curve connecting two points in a constant gravity field. Here, we change the objective to find the curve that minimizes the minimum distance from the curve to a point in space. The problem is first solved for the minimum-time objective and then a weighted objective method is used to scale down this component and focus on the minimum-distance cost functional. The problem is defined as follows.

$$\begin{aligned}
&\text{Min } J = w\big(\text{Min } (x-x_c)^2 + (y-y_c)^2\big) + (1-w)t_f \\
&\text{Subject to :} \\
&\dot{x} = v\cos(\Theta) \\
&\dot{y} = v\sin(\Theta) \\
&\dot{v} = g\sin(\Theta) \\
&\\
&t_0 = 0 \\
&x(t_0) = 0\,\text{m},\ y(t_0) = 0\,\text{m},\ v(t_0) = 1\,\text{m/s} \\
&x(t_f) = -20\,\text{m},\ y(t_f) = -20\,\text{m} \\
&\text{where} \\
&g = -9.81\,\text{m/s}^2,\ x_c = 7.5\,\text{m},\ y_c = -15\,\text{m}
\end{aligned}$$

The problem is solved using MATLAB's *bvp4c*. A continuation process was used, starting with a short trajectory and using each trajectory as the initial guess for the subsequent one.[5] The resulting trajectories and control histories are shown in Figure 2.

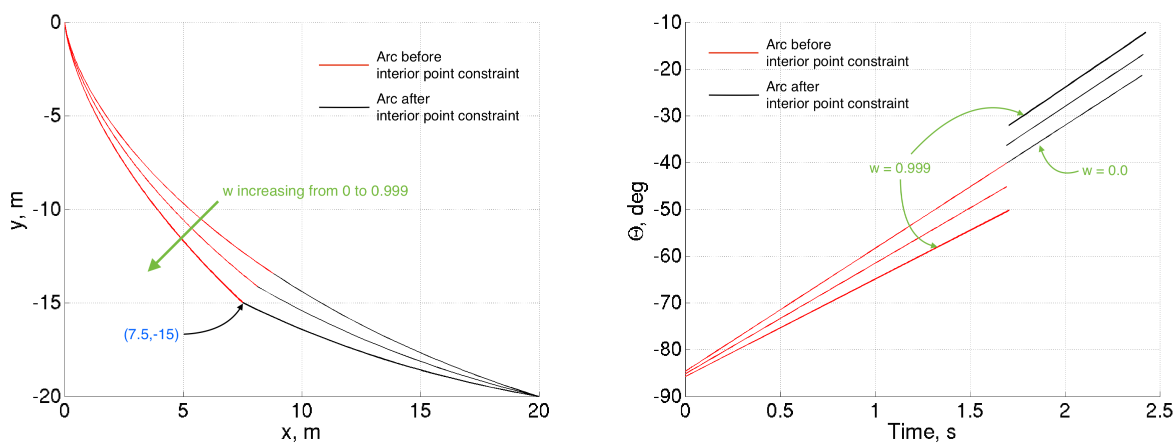


(a) y vs. x

(b) Control History

Figure 2. Solution for the original brachistochrone problem

The interior point constraint is introduced after this step, which splits the trajectory at the point where the distance to the specified point is the smallest. The weighting factor w in the objective functional is initially set to zero and it is modified in another series of continuation steps until the interior point cost functional forms 99.9% of the cost (Fig. 3). The change in the control as the weightage of the interior point cost is scaled up can be seen in Fig. 3b.



(a) y vs. x

(b) Control History

Figure 3. Solution for modified brachistochrone problem — Weighted objective function

The weighting factor cannot be set to 100% because the solution then becomes non-unique. There are an infinite number of curves connecting two points passing through a third point. Adding the minimum-time component ensures that the solution remains unique. It can be seen that the trajectory practically passes through the specified point $\mathbf{(7.5, -15)}$ when the weighting factor is set to 99.9%. The control history for a pure minimum time trajectory is the straight line in the middle of Fig. 3b corresponding to $w = 0.0$. As the interior point cost functional is factored in by changing w, the control can be seen to deviate drastically from the original profile.

B. Peak Heat-rate during Atmospheric Flight

An aerospace application of the interior point cost method is to minimize the peak heat rate of a hypersonic vehicle flying through an atmosphere using angle-of-attack control. This is an example of an objective functional for which only one time-derivative of the objective functional is required to obtain the control

($q = 1$). Therefore, according to Vinh, this cost functional can take the form of an isolated peak or a flat peak. Both these cases will be examined in this example, and the difference in the resulting trajectories will form the basis of the peak identification methodology described in Section IV.

In this scenario, a hypersonic gliding vehicle is released at a specified altitude and velocity by a boost vehicle. The trajectory of the gliding vehicle is initially optimized for maximum terminal velocity at a specified downrange position and altitude. The vehicle chosen has a mass of 340 kg and a peak L/D of around 2.5. A weighted objective method is used to perform a trade between maximum terminal velocity and minimum peak heat-rate solutions. The Sutton-Graves convective heating model[7] is used to compute the heating rate of the vehicle along the trajectory. The optimal control problem with 2D flight dynamics is defined as follows:

$$\begin{aligned}
&\text{Min } J = w\big(\text{Max } \dot{q}\big) + (1-w)v(t_f)^2 \\
&\text{Subject to} \\
&\qquad \frac{dh}{dt} = v\sin\gamma \\
&\qquad \frac{d\theta}{dt} = \frac{v\cos\gamma}{r} \\
&\qquad \frac{dv}{dt} = \frac{-D}{m} - \frac{\mu\sin\gamma}{r^2} \\
&\qquad \frac{d\gamma}{dt} = \frac{L}{mv} - \frac{\mu\cos\gamma}{vr^2} + \frac{v\cos\gamma}{r}
\end{aligned} \tag{8}$$

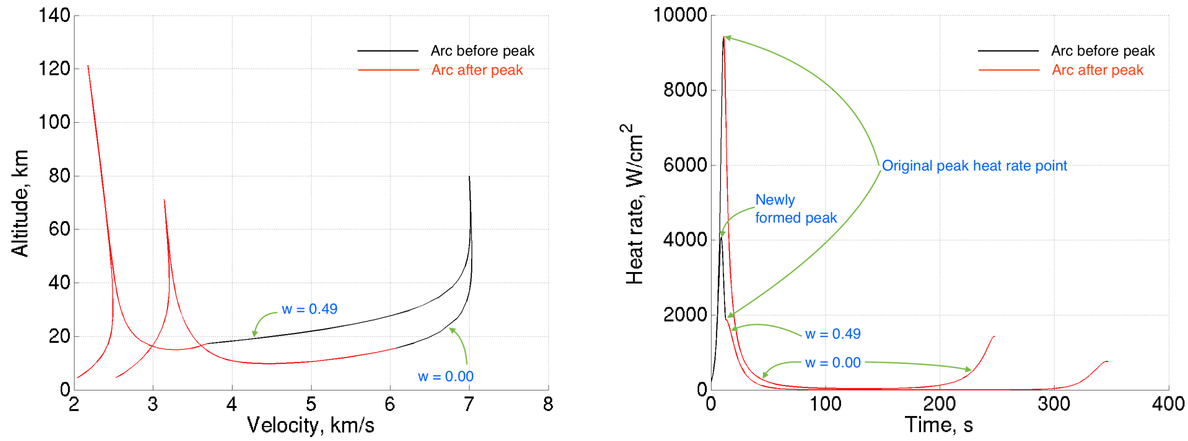
where

$$\begin{aligned}
&r = R_E + h \qquad \rho = \rho_0 e^{-h/H} \quad D = \frac{1}{2}\rho v^2 C_D A_{ref} \quad L = \frac{1}{2}\rho v^2 C_L A_{ref} \\
&\dot{q} = kv^3\sqrt{\frac{\rho}{r_n}}
\end{aligned} \tag{9}$$

Constraints in Eqs. (10) are used to impose the required post-boost and terminal conditions.

$$\begin{aligned}
h(0) &= 80\,\text{km} & h(t_f) &= 0\,\text{km} \\
\theta(0) &= 0^\circ & \theta(t_f) &= 7^\circ \\
v(0) &= 7\,\text{km/s} & v(t_f) &\text{ free} \\
\gamma(0) &= -60^\circ & \gamma(t_f) &\text{ free}
\end{aligned} \tag{10}$$

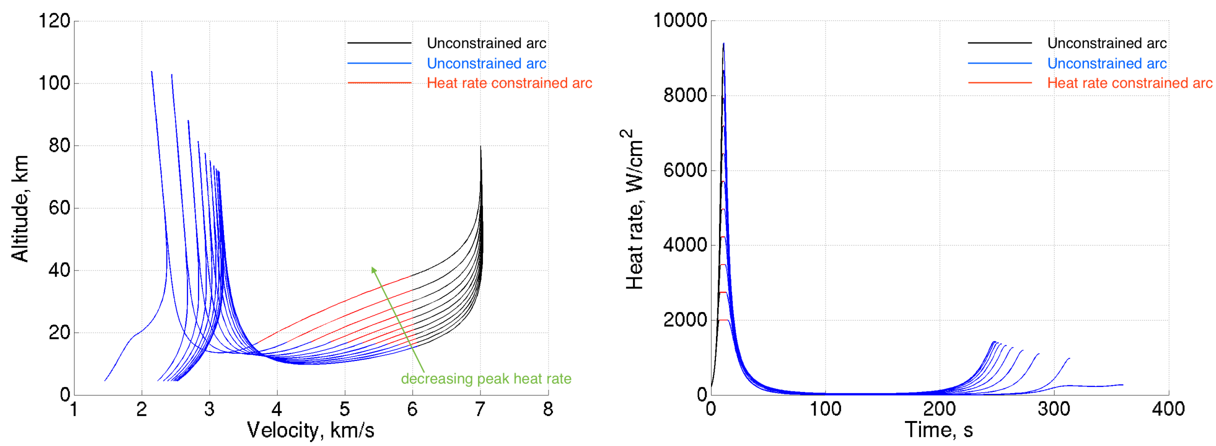
The problem is initially solved with the weighting factor (w) set to zero to obtain trajectories that maximize the terminal velocity for the given boundary conditions. The minimum peak heat rate cost functional is then introduced into this trajectory. This is done assuming the functional forms an isolated peak and then as a flat peak. The results are shown in Fig. 4 and Fig. 5 respectively.



(a) Altitude – Velocity

(b) Heat Rate vs. Time

Figure 4. Peak heat rate cost functional as an isolated peak.



(a) Altitude – Velocity

(b) Heat Rate vs. Time

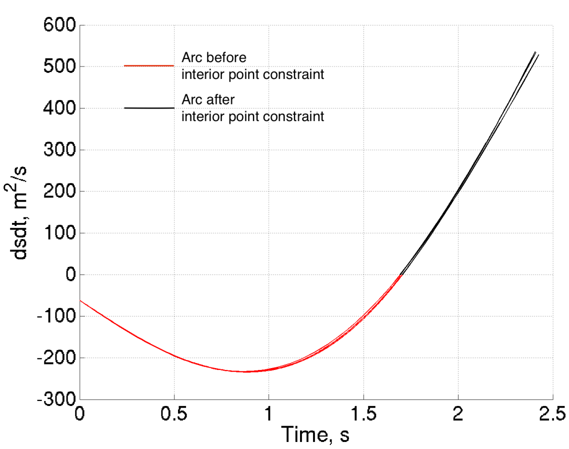
Figure 5. Peak heat rate cost functional as a flat peak.

It can be seen that solving the problem by assuming an isolated peak results in the trajectory changing such that new peaks in heat rate are formed as the constraint associated with the interior point cost functional is introduced at the initial peak. Alternatively, assuming that the cost functional is a flat peak, gives a uniform trajectory with no extra peak formation as shown in Fig. 5b. This behavior is the foundational underpinning of the automated peak identification methodology described in Section IV.

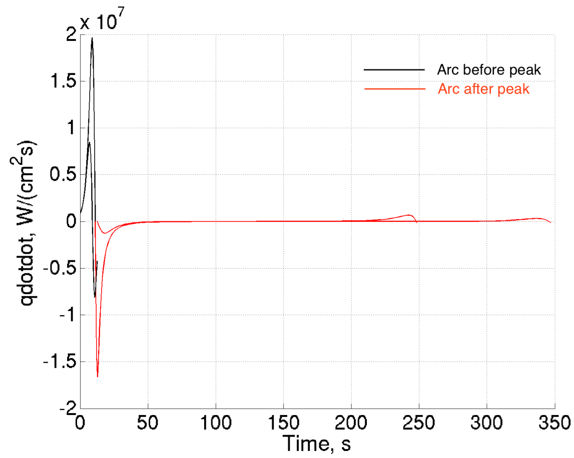
IV. Automated Identification of Peaks

An interior point cost functional with $q = 1$ can be optimized by considering it as having flat or isolated peaks. The type of peak to be used varies from problem to problem, and the right kind of peak is crucial to obtaining the optimal solution. A good example of this has been shown in Section III.B.

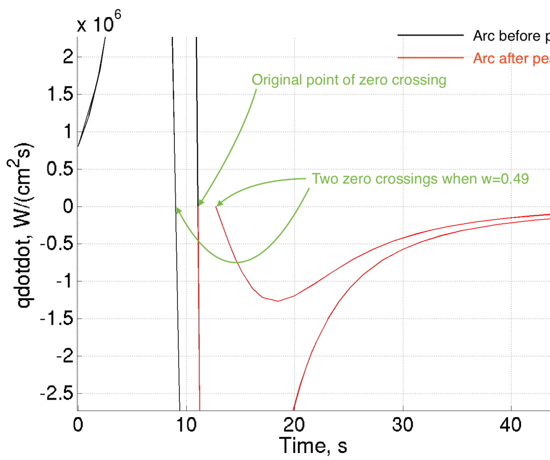
From the test scenarios, it was found that the numerical behavior of the cost functional around the peak determines whether the functional should be optimized as a flat peak or an isolated peak. One particular result of interest is the behavior of the time-derivative of the cost functional as it is introduced into the problem with a non-zero weighting factor (w). The automated peak identification method is based on numerical analysis of this behavior. The time-derivatives of the cost functionals from the two examples described above are shown in Fig. 6 when using the isolated peak method.



(a) Modified brachistochrone problem



(b) Peak heat rate problem



(c) Peak heat rate problem (magnified)

Figure 6. Time-derivative of the cost functional modeled as isolated peak.

For the brachistochrone problem, (Fig. 6a), a smooth time-derivative curve with a single sign change can be seen. This is a clear example of a single isolated peak in the cost functional. This behavior remains the same even after the interior point cost functional is introduced using the weighting factor. The curves mostly overlap and show the same trends even as the weighting factor is changed from 0 to 0.999. This seems to indicate that this particular cost functional is best modeled as an isolated peak. This also makes sense physically since the brachistochrone trajectory only has one point of closest approach to any given point in space.

The heat rate cost functional is also initially modeled as an isolated peak as shown in Fig. 4. The time-derivative of the cost functional for this case is plotted in Fig. 6b. The behavior of this curve is radically different from the derivative curve for the modified brachistochrone problem (Fig. 6a). As seen in Fig. 6c, before the interior point cost functional is introduced ($w = 0$), the curve has a single distinct zero-crossing event that corresponds to the peak in heat-rate at the 11 second mark.

As the interior point cost functional is introduced into the problem by changing the weighting factor, the cost functional curve is seen to form yet another higher peak (Fig. 4b) right before what was previously the highest peak. There is a corresponding second zero-crossing event formed in the time-derivative curve right before 10 seconds in Fig. 6c. Increasing the weighting factor does not seem to optimize the actual peak heat rate, but rather, decreases the heat rate at the point where the peak was at initially.

This behavior seems to be a strong indicator that peak heat-rate is a cost functional that is best optimized as a flat peak. This is further corroborated by the trajectory obtained by enforcing the cost functional as a flat peak in Fig. 5b. No extra peaks are formed and the the peak heat rate obtained this way is 2000 $\mathrm{W/cm^2}$, half of what was obtained from using the isolated peak method.

Based on these examples, it seems that the behavior of the time-derivative of the cost functional is a viable way of identifying the type of peak in an automated fashion using a simple numerical analysis. As mentioned before in Section II, the algorithm assumes that there is some other primary cost functional that the designer cares about which is optimized first before the interior point cost functional is introduced into the trajectory as a trade using a scalar weighting factor. This is required due to the nature of the continuation process used to solved the problem. The algorithm for solving "$q = 1$" interior point cost functionals with is described below

1. Solve problem without interior point cost function, using an alternative cost functional of interest.
2. Introduce interior point cost functional into the problem as a weighted objective functional assuming it to form an isolated peak.
3. Solve the new optimization problem with a small weighting in the interior point cost functional (e.g. $w = 0.01$) so that the new cost functional is just introduced into the problem.
4. Numerically evaluate the interior point cost functional over the entire trajectory and search for a peak higher than the one at the point where the cost functional was introduced. Supplement this with a search for any sign changes in the time-derivative of the interior point cost functional within a predefined distance to the peak.
5. Existence of a new peak after introduction of the cost functional along with a corresponding sign change in its derivative is a strong indicator that the cost functional *does not* form an isolated peak.
   - If the peak at the point where the cost functional is enforced remains the highest peak (as is the case in the Brachistochrone problem), the cost functional forms an isolated peak that can be solved without any further changes to the problem definition.
   - If a new, higher peak is detected, restart the optimization process from the original problem in Step 1 introducing the cost functional as a flat peak.

V. Conclusion

Minimax problems are those in which a cost functional is optimized at an intermediate point on the trajectory where the functional forms a peak. Original work performed by Vinh[1] identified the different types of peaks encountered when solving such problems. The number of time-derivatives (q) of the cost functional required for the control variable to appear forms the basis for the classification of these functionals. The cost functional forms isolated peaks when $q > 1$ and either flat or isolated peaks when $q = 1$. The method used to solve these problems depends on the type of peak and Vinh's work did not provide a way of distinguishing between them when $q = 1$.

A modified brachistochrone problem and a minimum peak heat-rate hypersonic trajectory problem were used to illustrate the two ways of optimizing "$q = 1$" cost functionals depending on the type of peak. The behavior of the time-derivative of the cost functional was shown to be a key factor in distinguishing between isolated peaks and flat peaks. Analysis of this behavior formed the basis for an automated strategy to distinguish between flat peaks and isolated peaks encountered in this type of interior point cost functional. Trying to optimize a flat peak as an isolated peak causes the formation of new peaks in the cost functional with corresponding zeros in its time-derivative. This is illustrated using the minimum peak heat-rate hypersonic trajectory problem. The brachistochrone problem illustrates the case where the cost functional does form an isolated peak. Using this general strategy, an algorithm for solving optimal control problems with interior point cost functionals was developed.

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